ELECTROSTATICS PART 1

Prerequisites

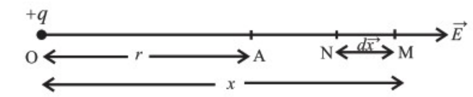
$$F = \frac{1}{4\pi\varepsilon}\frac{q_1 q_2}{r^2}$$

$$E = \frac{1}{4\pi\varepsilon}\frac{q}{r^2} \Rightarrow F = qE$$

$$W = \frac{1}{4\pi\varepsilon}\frac{q_1 q_2}{r} \Rightarrow W = \bar{F}.\overline{dr}$$

$$V = \frac{1}{4\pi\varepsilon}\frac{q}{r} \Rightarrow V = \frac{W}{Q}\ ; E = -\frac{dV}{dx}$$

Electric Potential due to point charge:



Consider a charge +q at O. We need to find electric potential at A, r away from O. Since potential is defined as the amount of work done in moving a unit positive charge from infinity to that point (here its ∞ to A).
Let's consider an intermediate point M, x away from O.

$$F = \frac{1}{4\pi\varepsilon}\frac{q}{x^2}$$

This force will be repulsive in nature. Now let us find the work done in moving the unit positive charge from M to N through an infinitesimal distance dx.

$dW = -Fdx,$

negative sign since displacement is oppositely directed to force

Hence total work done in bringing it from ∞ to A is

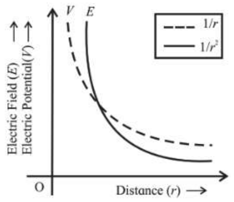
$$W = \int_{\infty}^{r} -Fdx = \int_{\infty}^{r} \frac{1}{4\pi\varepsilon}\frac{q}{x^2}dx = \frac{-q}{4\pi\varepsilon}\int_{\infty}^{r} x^{-2}dx$$

$$W = \frac{-q}{4\pi\varepsilon}\left[\frac{-1}{x}\right]_{\infty}^{r} = \frac{-q}{4\pi\varepsilon}\left[\frac{-1}{r} + \frac{1}{\infty}\right] = \frac{q}{4\pi\varepsilon r}$$

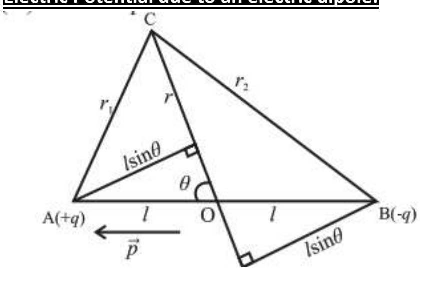
$$W = \frac{q}{4\pi\varepsilon_o r}\ for\ air$$

Thus, $V = W = \frac{q}{4\pi\varepsilon r}$

NOTE: V=0 at r=∞



Electric Potential due to an electric dipole:



Consider a dipole with charge +q and –q at A and B respectively. The line joining AB is the dipole axis.
Let C be a point r away from O (center of the dipole axis) and inclined at θ with the dipole axis. Let the distance of C from the charges at A and B be r_1 and r_2 respectively.
Potential at C due to A and B are

$$V_A = \frac{q}{4\pi\varepsilon_o r_1}\ and\ V_B = \frac{-q}{4\pi\varepsilon_o r_2}$$

The potential at C , $V_C = V_A + V_B$

$$V_C = \frac{q}{4\pi\varepsilon_o r_1} - \frac{q}{4\pi\varepsilon_o r_2} = \frac{q}{4\pi\varepsilon_o}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] \ldots\ldots (i)$$

By gemetry,

$$r_1^2 = (r - l\cos\theta)^2 + (l\sin\theta)^2$$
$$= r^2 - 2rl\cos\theta + l^2\cos^2\theta + l^2\sin^2\theta$$
$$r_1^2 = r^2 - 2rl\cos\theta + l^2$$
$$Similarly, r_2^2 = r^2 + 2rl\cos\theta + l^2$$

For a short dipole, where dipole length 2l << r

$$r_1^2 = r^2 - 2rl\cos\theta\ \ and\ r_2^2 = r^2 + 2rl\cos\theta$$

$$\frac{1}{r_1} = \frac{1}{\sqrt{r^2 - 2rl\cos\theta}}\ and\ \frac{1}{r_2} = \frac{1}{\sqrt{r^2 + 2rl\cos\theta}}$$

Substitute in (i) we get

$$V_c = \frac{q}{4\pi\varepsilon_o}\left[\frac{1}{\sqrt{r^2 - 2rl\cos\theta}} - \frac{1}{\sqrt{r^2 + 2rl\cos\theta}}\right]$$

$$V_c = \frac{q}{4\pi\varepsilon_o}\left[\frac{1}{\sqrt{r^2 - \frac{2r^2 l\cos\theta}{r}}} - \frac{1}{\sqrt{r^2 + \frac{2r^2 l\cos\theta}{r}}}\right]$$

$$V_c = \frac{q}{4\pi\varepsilon_o r}\left[\frac{1}{\sqrt{1 - \frac{2l\cos\theta}{r}}} - \frac{1}{\sqrt{1 + \frac{2l\cos\theta}{r}}}\right]$$

$$V_c = \frac{q}{4\pi\varepsilon_o r}\left[\left(1 - \frac{2l\cos\theta}{r}\right)^{-\frac{1}{2}} - \left(1 + \frac{2l\cos\theta}{r}\right)^{-\frac{1}{2}}\right]$$

$Since, \frac{2l\cos\theta}{r} \ll 1,$

using binomial expansion & retaining 1st order term

$$V_c = \frac{q}{4\pi\varepsilon_o r}\left[\left(1 - \left(-\frac{1}{2}\right)\frac{2l\cos\theta}{r}\right) - \left(1 + \left(-\frac{1}{2}\right)\frac{2l\cos\theta}{r}\right)\right]$$

$$V_c = \frac{q}{4\pi\varepsilon_o r}\left[\left(1 + \frac{l\cos\theta}{r}\right) - \left(1 - \frac{l\cos\theta}{r}\right)\right]$$

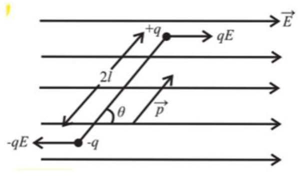
$$V_c = \frac{q}{4\pi\varepsilon_o r}\left(\frac{2l\cos\theta}{r}\right) = \frac{1}{4\pi\varepsilon_o}\frac{p\cos\theta}{r^2}\ where\ p = q.2l$$

$$At\ axis, \theta = 0^o\ or\ 180^o\ ,\qquad Vaxial = \pm\frac{1}{4\pi\varepsilon_o}\frac{p}{r^2}$$

$$At\ equator, \theta = 90^o\ , Veq = 0$$

Potential Energy of a dipole in an external Field:

Consider a dipole with charge +q and –q separated by a distance 2l, placed in a uniform electric field $\bar{E}$. It experiences a torque $\bar{\tau}$ which tends to rotate it.



$\bar{\tau} = \bar{p}\ X\ \bar{E}$, τ=pEsinθ

In order to neutralize it, lets apply an external torque $\overline{\tau_{ext}}$, which rotates it from angle θ_o to θ, without angular acceleration and an infinitesimal angular speed.

$$Word\ done\ = W = \int_{\theta_o}^{\theta}\bar{\tau}_{ext}\ d\theta = \int_{\theta_o}^{\theta} pE\sin\theta\ d\theta = pE[-\cos\theta]_{\theta_o}^{\theta}$$

$$W = pE[-\cos\theta - (-\cos\theta_o)] = pE[\cos\theta_o - \cos\theta].$$

This work is stored as Potential Energy (U)

CASE 1: Initially dipole is perpendicular θ_o=π/2

U = pE[cos π/2 – cos θ] = –pEcosθ = $-\bar{p}.\bar{E}$

CASE 2: Initially the dipole is parallel to the field θ_o=0

U = pE [cos 0 – cos θ] = **pE (1 – cosθ)**